Algebraic structures: groups ***G***, semigroups ***S***, fields ***F***, F_p, Z_p, rings ***R***, Z_{p-1}.

$<G, *>$	$<S, *>$	$<F, \bullet, +>$	$<F, \bullet, +>$
$<Z_p^*, \bullet>$	$<Z_{p-1}, \bullet>$	$<Z_p, \bullet, +>$	$<Z_{p-1}, \bullet, +>$
	$<Z_{p-1}, +>$		

$\mathcal{Z}_p^* = \{1, 2, 3, \ldots, p-1\}$ $\mathcal{I}_{p-1} = \{0, 1, 2, 3, \ldots, p-2\}$ $\bullet \bmod p-1$; $+ \bmod (p-1)$

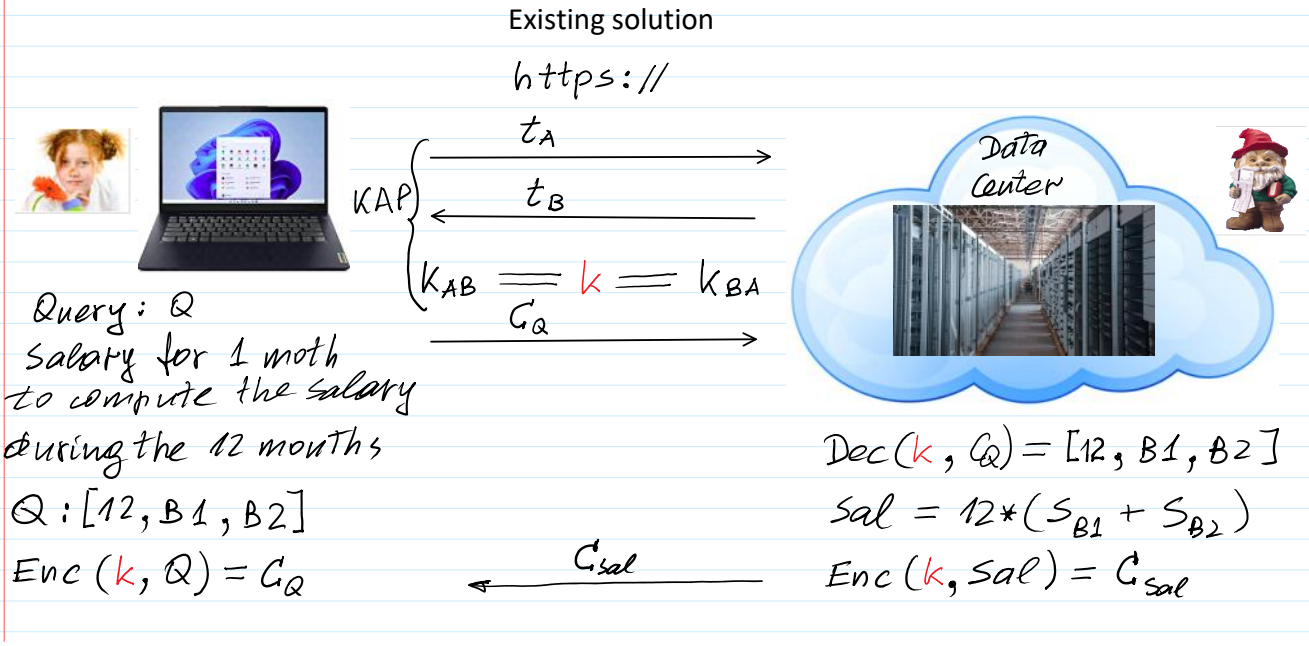
Commutative algebraic structures

Algebraic strukture	Number of binary operations	Multiplicative operation •	Inverse operation /	Additive operation +	Inverse operation -	Distributivity property a· (b+c) = a· b+a· c	Examples
Multiplicative **Semigroup**	1	Yes	No	No	No	No	$<Z_n, \bullet>$ • mod n
Additve **Semigroup**	1	No	No	Yes	No	No	**Set of reals N**
Multiplicative **Group**	1	Yes	Yes	No	No	No	$<Z_p^*, \bullet>$
Additve **Group**	1	No	No	Yes	Yes	No	$<Z_n, +>$ + mod n
Ring	2	Yes	No	Yes	Yes	Yes	$<Z_n, \bullet, +>$
Field	2	Yes	Yes	Yes	Yes	Yes	$<Z_p, \bullet, +>$

DEF : $\mathcal{I}_{p-1} \to \mathcal{Z}_p^*$, where p - is prime.

DEF : is defined by public parameter $PP = (p, g)$

Homomorphic CryptoSystems: Computation with encrypted data in Data Center.



$Dec(k, C_{sal}) = sal$

Cloud services

$Enc(k, B1) = C_{B1}$

$Enc(k, B2) = C_{B2}$

$Enc(k, 12) = C_{12}$

$C_{B1}, C_{B2}, C_{12} \rightarrow$

$\leftarrow C_S$

$C_S = C_{12} * (C_{B1} + C_{B2})$

$Dec(k, C_S) = sal$

$sal = 12 * (B1 + B2)$

Homomorphic encryption

$F : S_1 \rightarrow S_2$; $F \in \{surjective, injective, \overset{1-to-1}{bijective}\}$

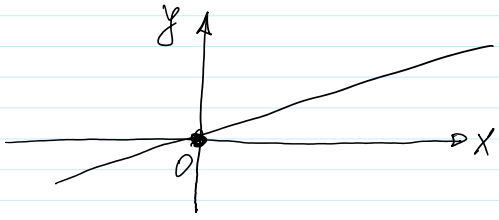
$< S_1, * >$, $< S_2, \bullet >$

$\forall x, y \in S_1 : F(x * y) = F(x) \bullet F(y)$ where $F(x) = a$, $F(y) = b$, $a, b \in S_1$.

If mapping F is bijective (1-to-1) then homomorphism F is isomorphism

Let R be a field of reals: $< R, \cdot, + >$

$f : R \rightarrow R$; $f(x) = kx$



1. check: $f(x_1 + x_2) = k(x_1 + x_2) = kx_1 + kx_2 = f(x_1) + f(x_2)$

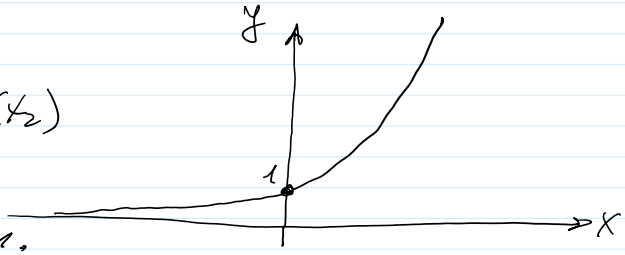
2. check: $f(x_1 \cdot x_2) = k \cdot x_1 \cdot x_2 \neq kx_1 \cdot kx_2 = k^2 \cdot x_1 \cdot x_2$

3. check: $f(x) = e^x$; $f : R \rightarrow R^+$

$f(x_1 + x_2) = e^{x_1 + x_2} = e^{x_1} \cdot e^{x_2} = f(x_1) \cdot f(x_2)$

$< R, + > \rightarrow < R^+, \cdot >$

Additively – multiplicative isomorphism.



3) check. $f(x) = g^x \bmod p$; $PP = (p, g)$

Ferma theorem: If $x \neq 0$, then for $\forall x \in \mathbb{Z}_{p-1}$

$x^{p-1} = 1 \bmod p$

$f : \mathbb{Z}_{p-1} \rightarrow \mathbb{Z}_p^*$ and is 1-to-1 (bijective) mapping.

$x^{\cdot} = 1 \bmod p$

$f: \mathbb{Z}_{p-1} \to \mathbb{Z}_p^*$ and is 1-to-1 (bijective) mapping.

$f(x_1 + x_2) = g^{x_1+x_2} \bmod p = g^{x_1} \cdot g^{x_2} \bmod p = f(x_1) \cdot f(x_2) = a \cdot b \bmod p.$

Additively - multiplicative homomorphism.

4 check. $f_i(m) = m\, g^i \bmod p$

$f_i(m_1 + m_2) = (m_1 + m_2)\, g^i \bmod p = m_1 g^i \bmod p + m_2 g^i \bmod p = f_i(m_1) + f_i(m_2)$

$f_i : \mathbb{Z}_p^* \to \mathbb{Z}_p^*.$

5. check. $f_{i,g}(m) = g^m \cdot g^i \bmod p$

$f_{i,g}(m_1 + m_2) = g^{m_1+m_2} \cdot g^i \bmod p = g^{m_1} \cdot g^{m_2} \cdot g^i \bmod p \neq f_{i,g}(m_1) \cdot f_{i,g}(m_2)$

6. check. $f_{i_1,g}(m_1) = g^{m_1} \cdot g^{i_1} \bmod p$; $f_{i_2,g}(m_2) = g^{m_2} \cdot g^{i_1} \bmod p$

$f_{i,g}(m_1 + m_2) = g^{m_1+m_2} \cdot g^i \bmod p = g^{m_1} \cdot g^{m_2} \cdot g^i \bmod p =$

$i = (i_1 + i_2) \bmod (p-1)$

$= g^{m_1} \cdot g^{m_2} \cdot g^{i_1+i_2} \bmod p = g^{m_1} \cdot g^{i_1} \cdot g^{m_2} \cdot g^{i_2} \bmod p$

$= f_{i_1,g}(m_1) \cdot f_{i_2,g}(m_2).$

Additively - multiplicative isomorphism if $i = (i_1 + i_2) \bmod (p-1)$.

If p - is strong prime, then $p = 2q + 1$, where q - is prime as well.

An element $g \in \mathbb{Z}_p^*$ is a generator in $\mathbb{Z}_p^*$

```
>> p=genstrongprime(28)
p = 232702259
>> pb=dec2bin(p)
pb = 1101 1101 1110 1100 0001 0011 0011
```

if and only if: 1) $g^q \neq 1 \bmod p$

2) $g^2 \neq 1 \bmod p$

```
>> g=2
g = 2
>> mod_exp(g,q,p)
ans = 232702258
>> mod_exp(g,2,p)
ans = 4
```

$g^{m_1} \cdot g^{m_2} = g^{(m_1+m_2) \bmod (p-1)} \bmod p$

$n_{12} = n_1 \cdot n_2 \bmod p = g^{(m_1+m_2) \bmod (p-1)} \bmod p$

```
>> m1=2000
m1 = 2000
>> m2=3000
m2 = 3000
>> n1=mod_exp(g,m1,p)
n1 = 228510651
>> n2=mod_exp(g,m2,p)
n2 = 220266692
>> n12=mod(n1*n2,p)
n12 = 181510254
```

```
>> m1pm2=mod(m1+m2,p-1)
m1pm2 = 5000
>> em1pm2=mod_exp(g,m1pm2,p)
em1pm2 = 181510254
```

Additively - multiplicative homomorphism.

Additively – multiplicative homomorphism.